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BOUNDARY LAYERS IN WEDGES OF LAMINATED COMPOSITE STRIPS UNDER GENERALIZED PLANE DEFORMATION—PART I: ASYMPTOTIC SOLUTIONS

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Abstract—Based upon Lekhnitskii's formulation and the Stroh formalism, the structure of the asymptotic solutions has been examined for the boundary layer on the wedge type cross section of a laminated composite strip. The composite strip is assumed under the so-called generalized plane deformation, which includes tension, bending and/or torsion by the terminal tractions as well as the generalized plane strain problem. The solution structures are obtained, with the aid of numerical calculation, for various kinds of wedge geometry including the free edge and the delamination cracks with the crack faces opened or closed. Finally the nature of the asymptotic solutions is discussed, including the mode mixity of singular stress field ahead of the wedge tip, it is found that for a free edge problem the mode mixity of the singular asymptotic traction vector on the interfacial plane near the free edge remains invariant under varying types of remote loadings once a pair of adjacent materials (or ply orientations) is given and that accordingly one single scaling parameter governs the near field response.

1. INTRODUCTION

Deformations of a laminated composite inherently involve a boundary layer region on which the deformation field is locally distorted owing to material or geometric discontinuity such as ply interfaces, free edges, cracks or cut-outs. In such a boundary layer, there are often found local stress singularities and inherently three-dimensional states of complex stresses. Moreover, the high local stresses and associated deformations caused by these geometric and material discontinuities often result in undesirable delamination and transverse crack initiation and growth, leading to final fracture. Therefore, development of an analytical method that can provide insight into a boundary layer region is of foremost importance to the analysts. The interfacial or transverse crack problems and free edge problems are among typical examples concerning boundary layers in mechanics of composite laminates, and they have been among the subjects under intensive investigation during the last two decades.

There are two lines of approach to plane problems in the theory of anisotropic elasticity. Lekhnitskii (1963) introduced the complex potentials for stress to treat the generalized plane deformation problems. He applied the compatibility equations ultimately to lead to a set of coupled elliptic partial differential equations for the complex potentials, and relied upon the complex characteristics for the elliptic partial differential equation to obtain the general representation for solutions. On the other hand, Eshelby *et al.* (1953) and Stroh (1962) began with a simple representation for displacement fields finally to reach the so-called Stroh formalism for generalized plane strain problems. The Stroh formalism, adopted by researchers in the material science community, has been applied mainly to the generalized plane strain problems including plane strain or plane stress problems related to dislocations and interfaces in the crystals.

Wang and Choi (1982) employed Lekhnitskii's formulation to obtain the solution for a free edge problem under extension. Subsequently Wang (1984) obtained the solution for a delamination crack under extension in a similar method. In these works, the particular solutions for stress were assumed to be polynomial functions in the beginning, and finally reduced to a constant for uniform extension. On the other hand, Zwiers *et al.* (1982) made a finding based upon the Stroh formalism (Stroh, 1962): the possible existence of logarithmic particular solutions for stress in free edge problems under uniform extension. The uniform extension treated by these authors is one of the most fundamental deformations; however, it occurs, in general, only for a laminated composite with no elastic couplings: curvature and twist as well as extension will occur even under the uniaxial tension of a laminated composite with non-zero coupling compliance. Moreover, the results from Zwiers *et al.* (1982) suggest that particular solutions for stress may include logarithmic functions under curvature or twist as well as under the uniform extension. Unfortunately, neither complete asymptotic solutions (except those for the uniform extension) nor the nature of the solution have been reported to date, although the solution is of paramount importance in association with the understanding of fundamental fracture behavior of composite laminates under more generic loadings.

In this paper, we examine the asymptotic solutions (homogeneous and particular solutions) near the boundary layer regions on a wedge type cross-section of a laminated composite strip under generalized plane deformation. For this we rely upon Lekhnitskii's representation for displacements, and the Stroh formalism, which has been found to be useful for calculating the asymptotic solution for the free edge problem under uniform extension (Zwiers *et al.*, 1982). This solution procedure is extended here to the general cases of deformation involving curvature, twist, as well as extension so that the uniaxial tension or compression, pure bending and torsion or a combination of these may be treated in terms of loading. Moreover the nature of the asymptotic solution including the mode mixity is discussed. The complete numerical solution for various wedges including delamination cracks is reported separately in the associated paper (Kim and Im, 1994).

In Section 2.1, the problem under consideration is described, and then based upon Leknitskii's formulation and the Stroh formalism under generalized plane deformation, the solution form for stress and displacement field is obtained, and the asymptotic solutions from the eigenfunction expansion are presented. In Section 2.2, the near field conditions for wedge problems are extended to lead to the structure of solutions, which consist of homogeneous and particular solutions for stress and displacement. The asymptotic form of homogeneous solutions for stress and displacement, including the stress singularity, is determined in Section 3. In Section 4, the existence of the polynomial type particular solutions in a composite wedge under generalized plane deformation, subjected to the aforementioned generic loadings, is investigated, and numerical results and discussion for particular solutions are followed in Section 5.1. In Section 5.2, some results are discussed regarding the so-called mode mixity for a singular stress field near a wedge, which is a measure of the ratio of the singular shear stress to the singular normal stress ahead of a wedge tip: it is found that for certain cases, the mode mixity is independent of the loading or the remote boundary conditions. Finally concluding remarks are made in Section 6.

2. FORMULATION OF THE PROBLEM

2.1. *Generalization of the plane deformation problem*

Consider a laminated composite strip, the cross-section of which contains wedges, subjected to general end loadings such as uniaxial tension, pure bending and/or torsion (see Fig. 1). Each ply of the composite laminates lies in a plane parallel to the x_1–x_3 plane, and the ply orientation is defined to be the counter-clockwise angle, viewed from the top, that the fiber direction makes with the x_3-axis. We assume that the laminate dimension in the x_3 direction (laminate length) is sufficiently large compared with the laminate thickness so that the state of strains and stresses depends upon only the two coordinates x_1 and x_2 under the aforementioned loadings; the laminate is then said to be in the state of generalized plane deformation on the x_1–x_2 plane. This class of deformations includes also the well-known generalized plane strain deformation under lateral tractions that do not vary along the laminate length. Note that the displacement in the direction of the laminate length may not disappear in such a generalized plane strain deformation although the normal strain in this direction is identically zero; only when there exists appropriate material symmetry,

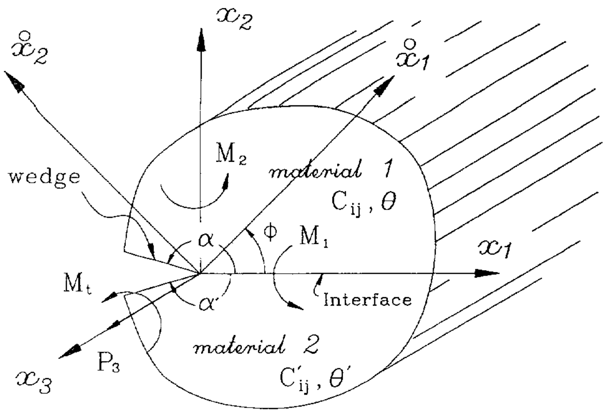

Fig. 1. Composite wedge under generic loadings.

both the displacement and normal strain in the x_3 direction will disappear and generalized plane strain deformation be reduced to a plane strain deformation.

Let u_i, ε_{ij}, σ_{ij} denote the Cartesian components of displacement, strain and stress, respectively. For generalized plane deformation, we have the equilibrium equation, strain–displacement relation and stress–strain relation:

$$\sigma_{i\beta,\beta} = 0 \quad (i = 1\text{–}3, \beta = 1,2) \quad \text{(no body force)}, \tag{1a}$$

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \tag{1b}$$

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km}, \quad C_{ijkm} = C_{jikm} = C_{kmij}, \tag{1c}$$

where C_{ijkm} are 4th order stiffness tensors, and the comma indicates the partial differentiation with respect to x_i. Note that the aforementioned governing equations and all relevant equations to follow, in principle, have to be written for each of the two adjacent plies at a wedge, and we employ the expressions without "prime" for the upper ply and the expressions with prime for the lower ply. We state the expressions only for the upper ply whenever we can deduce the expressions for the lower ply from those for the upper ply. We suppose the loadings of uniaxial tension, pure bending and torsion, only, which result in the state of generalized plane deformation in the composite strip, and make the problem two dimensional.

When the aforementioned loadings are applied to the ends of a composite strip and when six rigid body modes are neglected, from the constitutive equations and compatibility relation the displacement component u_i can be obtained as in Lekhnitskii (1963):

$$u_i(x_1, x_2, x_3) = U_i(x_1, x_2) + \delta_{i1}(-A_2x_3^2/2 - A_4x_2x_3) + \delta_{i2}(-A_3x_3^2/2 + A_4x_1x_3) + \delta_{i3}(A_2x_1 + A_3x_2 + A_1)x_3, \tag{2}$$

where δ_{ij} is the Kronecker delta; A_1 is a deformation parameter related to axial extension along the x_3-axis; A_2 and A_3 are parameters related to curvatures in the x_1–x_3 planes and the x_2–x_3 planes, respectively; and A_4 is the parameter related to twist along the x_3-axis. With the aid of eqns (1b,c) and (2), we can show that equilibrium equation (1a) leads to

$$(C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) + C_{i1k1}\frac{\partial^2 U_k}{\partial x_1^2} + (C_{i1k2} + C_{i2k1})\frac{\partial^2 U_k}{\partial x_1 \partial x_2} + C_{i2k2}\frac{\partial^2 U_k}{\partial x_2^2} = 0. \tag{3}$$

The general solution for $U_i(x_1, x_2)$ in eqn (3) is represented as the sum of the homogeneous and particular solutions

$$U_i(x_1, x_2) = U_i^{\mathrm{h}}(x_1, x_2) + U_i^{\mathrm{p}}(x_1, x_2), \tag{4}$$

and U_i^{h} and U_i^{p} satisfy the following equation, respectively,

$$C_{i1k1}\frac{\partial^2 U_k^{\mathrm{h}}}{\partial x_1^2} + (C_{i1k2} + C_{i2k1})\frac{\partial^2 U_k^{\mathrm{h}}}{\partial x_1 \partial x_2} + C_{i2k2}\frac{\partial^2 U_k^{\mathrm{h}}}{\partial x_2^2} = 0, \tag{5a}$$

$$C_{i1k1}\frac{\partial^2 U_k^{\mathrm{p}}}{\partial x_1^2} + (C_{i1k2} + C_{i2k1})\frac{\partial^2 U_k^{\mathrm{p}}}{\partial x_1 \partial x_2} + C_{i2k2}\frac{\partial^2 U_k^{\mathrm{p}}}{\partial x_2^2} + (C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) = 0. \tag{5b}$$

Note that eqn (5a), obtained when the deformation parameters A_i ($i = 2, 3, 4$) become zero, is just the governing equation for the generalized plane strain deformations. Moreover, this is an elliptic type of partial differential equation, characteristics of which are complex. In terms of the complex characteristics, the homogeneous solution U_i^{h} may be written as (Stroh, 1962; Ting, 1986; or Suo, 1990):

$$U_i^{\mathrm{h}}(x_1, x_2) = \sum_{k=1}^{6} v_{ik} f(z_k), \quad z_k = x_1 + \mu_k x_2 \quad (k = 1\text{–}6), \tag{6}$$

where μ_k are complex or purely imaginary eigenvalues to be determined, and $f(z_k)$ is a function of complex characteristics z_k. Substituting eqn (6) into eqn (5a), we obtain the equations:

$$M_{ij}(\mu_k) v_{jm} = 0, \quad (m = k)$$

where

$$M_{ij}(\mu_k) = C_{i1j1} + \mu_k(C_{i1j2} + C_{i2j1}) + \mu_k^2 C_{i2j2}.$$

For the existence of nontrivial solutions, we have

$$\det[M_{ij}] = |M_{ij}| = 0,$$

and the solutions of this sextic equation yield three pairs of complex conjugate eigenvalues (Stroh, 1962; Ting, 1986; or Suo, 1990),

$$\mu_k = \bar{\mu}_{k+3}, \quad (k = 1, 2, 3),$$

and three pairs of associated eigenvectors v_{ik} can now be obtained through a proper normalization and satisfy the following equation

$$v_{i(k+3)} = \bar{v}_{ik}, \quad (k = 1, 2, 3).$$

In this work we assume that the eigenvalues μ_k are distinct. Discussion of the cases wherein μ_k has a multiple root can be found in Ting and Chou (1981).

As the first step towards obtaining the asymptotic solution, we rely upon the power type eigenfunction for $f(z_k)$ as given by Wang and Choi (1982) and Ting (1986)

$$f(z_k) = \sum_{n=1}^{\infty} C_{kn} z_k^{\delta_n+1}/(\delta_n+1) \quad (k = 1\text{–}6), \tag{7}$$

which leads to the expression for the homogeneous solution

$$U_i^h(x_1, x_2) = \sum_{n=1}^{\infty} \sum_{k=1}^{3} [C_{kn} v_{ik} z_k^{\delta_n+1} + C_{(k+3)n} \bar{v}_{ik} \bar{z}_k^{\delta_n+1}]/(1+\delta_n), \tag{8}$$

where an overbar denotes the complex conjugate; C_{kn} and $C_{(k+3)n}$ are complex constants, and the subscript k means the three pairs of eigenvalues. Here the eigenvalues δ_n are to be determined from homogeneous equation of the so-called "near field" conditions, including the traction conditions and interface continuity conditions. The coefficient C_{kn} is dependent upon the associated eigenvalues δ_n, and it can be determined within an arbitrary constant when δ_n is obtained.

From the structure of the partial differential eqn (5b), the particular solutions for the displacement U_i^p may be written in a generic complete quadratic polynomial. However, this will involve total 30 unknown constants (15 for each ply) for the strip at hand when rigid body modes for the strip are neglected, and therefore it would entail extremely complex algebra. For a systematic approach to determining the particular solutions U_i^p, we may take advantage of the type of expression as given by eqn (8), which satisfies the equilibrium eqn (5a) identically. Careful examination shows that we can represent a generic quadratic polynomial, with neglect of rigid body modes, in the following form:

$$U_i^p(x_1, x_2) = \sum_{k=1}^{3} [D_{k0} v_{ik} z_k + \bar{D}_{k0} \bar{v}_{ik} \bar{z}_k] + \frac{1}{2} \sum_{k=1}^{3} [D_{k1} v_{ik} z_k^2 + \bar{D}_{k1} \bar{v}_{ik} \bar{z}_k^2] + \tfrac{1}{2}\lambda_i x_1^2 \tag{9}$$

where D_{k0} and D_{k1} are complex constants to be determined and λ_i are real. Note that the addition of an arbitrary single quadratic term, like the last term $\frac{1}{2}\lambda_i x_1^2$ above, makes the right hand side of eqn (9) a complete quadratic polynomial of 15 terms for one ply; thus there are total $2 \times 15 = 30$ terms for the composite wedge of the strip with neglect of the six terms associated with rigid translation, for there are six complex terms from D_{k0} and D_{k1}, and three real terms λ_i in eqn (9). Then the three real constants λ_i are now determined from substitution of eqn (9) into eqn (5b):

$$(C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) + C_{i1k1}\lambda_k = 0. \tag{10}$$

The complex constants D_{k0} and D_{k1} will be determined from the non-homogeneous equations resulting from the near field conditions. The aforementioned approach is relatively simple, and moreover systematic in that it will provide a clue for introducing a logarithmic function (Zwiers *et al.*, 1982), as will be shown in Section 4.2, when the polynomial form of eqn (9) does not work.

For the convenience of further development, we introduce the cylindrical coordinate (r, ϕ, z) to write $z_k = x_1 + \mu_k x_2$ in r and ϕ,

$$z_k = x_1 + \mu_k x_2 = r\zeta_k, \quad \zeta_k = \cos\phi + \mu_k \sin\phi. \tag{11}$$

The physical vector or tensor components in the cylindrical coordinate (r, ϕ, z) are equivalent to the corresponding components in the Cartesian co-ordinate system $(\mathring{x}_1, \mathring{x}_2, x_3)$ which is obtained by rotating the coordinate system (x_1, x_2, x_3) counterclockwise by ϕ (Fig. 1), i.e. $\mathring{u}_1 = u_r$, $\mathring{u}_2 = u_\phi$, $\mathring{u}_3 = u_z$, $\mathring{\sigma}_{12} = \sigma_{r\phi}$ etc. where "°" indicates the components referred to the rotated Cartesian coordinate system $(\mathring{x}_1, \mathring{x}_2, x_3)$. We hereafter use these two notations interchangeably for convenience. Now the displacement and stress components in the Cartesian coordinate $(\mathring{x}_1, \mathring{x}_2, x_3)$ (or in the cylindrical coordinate) can be written as:

$$\mathring{u}_i(r,\phi,x_3) = \mathring{U}_i^{\mathrm{h}}(r,\phi)+\mathring{U}_i^{\mathrm{p}}(r,\phi)+\mathring{u}_i^{\mathrm{A}}(r,\phi,x_3) = \mathring{u}_i^{\mathrm{h}}(r,\phi)+\mathring{u}_i^{\mathrm{p}}(r,\phi,x_3), \tag{12a}$$

$$\mathring{\sigma}_{ij}(r,\phi) = \mathring{\sigma}_{ij}^{\mathrm{h}}(r,\phi)+\mathring{\sigma}_{ij}^{\mathrm{p}}(r,\phi), \tag{12b}$$

where

$$\mathring{u}_i^{\mathrm{h}}(r,\phi) = \mathring{U}U_i^{\mathrm{h}}(r,\phi), \quad \mathring{u}_i^{\mathrm{p}}(r,\phi,x_3) = \mathring{U}_i^{\mathrm{p}}(r,\phi)+\mathring{u}_i^{\mathrm{A}}(r,\phi,x_3),$$

and the expressions for $\mathring{U}_i^{\mathrm{h}}$, $\mathring{U}_i^{\mathrm{p}}$, $\mathring{u}_i^{\mathrm{A}}$, $\mathring{\sigma}_{ij}^{\mathrm{h}}$, $\mathring{\sigma}_{ij}^{\mathrm{p}}$ are given in Appendix A. These expressions may be written for each ply of the laminate, for example, the upper ply and the lower ply: in which case we use the above expression for the upper ply, and the "primed" expressions for the lower ply.

2.2. *Solution structure under generalized plane deformation*

To determine the structure of the asymptotic solutions including the stress singularities, we need to consider the near field conditions for the composite wedge in Fig. 1. Assuming the two plies are perfectly bonded along the interface, the near field conditions may be written as: traction condition: $\sigma_{r\phi} = \sigma_{\phi\phi} = \sigma_{\phi z} = 0$ on $\phi = \alpha$, $\sigma'_{r\phi} = \sigma'_{\phi\phi} = \sigma'_{\phi z} = 0$ on $\phi = \alpha'$ if wedge faces are opened, or $\sigma_{r\phi} = \sigma_{\phi z} = \sigma'_{r\phi} = \sigma'_{\phi z} = 0$, $[u_\phi] = [\sigma_{\phi\phi}] = 0$ on $\phi = \pm\pi$ for a crack if crack faces are in frictionless contact. The interface continuity condition is $[\sigma_{r\phi}] = [\sigma_{\phi\phi}] = [\sigma_{\phi z}] = 0$, $[u_r] = [u_\phi] = [u_z] = 0$ on $\phi = 0$ where [ ] indicates the discontinuity of the quantity in it across the ply interface. Substituting (12a,b) for displacement and stress into the near field conditions, we obtain a system of 12×12 linear equations for C_{kn}, $C_{(k+3)n}$, C'_{kn}, and $C'_{(k+3)n}$ or $D_{k\alpha}$, $\bar{D}_{k\alpha}$, $D'_{k\alpha}$ and $\bar{D}'_{k\alpha}$ ($k = 1$–3, $\alpha = 0, 1$), which can be written as

$$\sum_{n=1}^{\infty} r^{\delta_n}\mathbf{K}_c(\delta_n)\mathbf{q} = A_1\mathbf{b}_1+r(A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4), \tag{13}$$

where $\mathbf{K}_c$ is a complex valued square matrix whose elements depend upon δ_n, and $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$ are constant column matrices related to material constants and the near field conditions, and $\mathbf{q}$ is a 12×1 column matrix whose elements are C_{kn}, $C_{(k+3)n}$, C'_{kn}, and $C'_{(k+3)n}$, or $D_{k\alpha}$, $\bar{D}_{k\alpha}$, $D'_{k\alpha}$ and $\bar{D}'_{k\alpha}$ ($k = 1$–3, $\alpha = 0, 1$). The detailed expressions for $\mathbf{K}_c(\delta_n)$ and $\mathbf{b}_i$ ($i = 1$–4) in terms of wedge angles and material properties are listed in the Appendix B. To satisfy eqn (13), we let $\delta_n = 0$ and $\delta_n = 1$:

$$\mathbf{K}_c(0)\mathbf{q}^{\mathrm{P}_0} = A_1\mathbf{b}_1, \quad \mathbf{K}_c(1)\mathbf{q}^{\mathrm{P}_1} = A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4, \tag{14a,b}$$

and obtain two particular solutions $\mathbf{q}^{\mathrm{P}_0}$ and $\mathbf{q}^{\mathrm{P}_1}$, where $\mathbf{q}^{\mathrm{P}_0}$ and $\mathbf{q}^{\mathrm{P}_1}$ indicate the column vectors

$$\mathbf{q}^{\mathrm{P}_0} = [D_{10}, D_{20}, D_{30}, \bar{D}_{10}, \bar{D}_{20}, \bar{D}_{30}, D'_{10}, D'_{20}, D'_{30}, \bar{D}'_{10}, \bar{D}'_{20}, \bar{D}'_{30}]^{\mathrm{T}},$$

$$\mathbf{q}^{\mathrm{P}_1} = [D_{11}, D_{21}, D_{31}, \bar{D}_{11}, \bar{D}_{21}, \bar{D}_{31}, D'_{11}, D'_{21}, D'_{31}, \bar{D}'_{11}, \bar{D}'_{21}, \bar{D}'_{31}]^{\mathrm{T}}.$$

This is, however, not the only solution for $\mathbf{q}$ in eqn (13). We see that $\mathbf{q}$ has the following homogeneous solution

$$\mathbf{K}_c(\delta_n)\mathbf{q}^{\mathrm{h}} = 0, \tag{15}$$

where

$$\mathbf{q}^{\mathrm{h}} = [C_{1n}, C_{2n}, C_{3n}, C_{4n}, C_{5n}, C_{6n}, C'_{1n}, C'_{2n}, C'_{3n}, C'_{4n}, C'_{5n}, C'_{6n}]^{\mathrm{T}}.$$

Now the substitution of $\mathbf{q}^{\mathrm{h}}$, $\mathbf{q}^{\mathrm{P}_0}$ and $\mathbf{q}^{\mathrm{P}_1}$ into eqn (8) and eqn (9), respectively, gives the solution for U_i^h and U_i^p, and eqns (12a,b) then yield the complete solution structure for the asymptotic displacement and stress. It is noted that the particular solution $\mathbf{q}^{p_0}$ is related to the uniform extension with neither of curvatures and twist, wherein only A_1 is nonzero, and this case was reported by Zwiers *et al.* (1982). On the other hand, the particular solution $\mathbf{q}^{p_1}$, related to the parameters A_2, A_3 and A_4, is associated with curvatures and twist. Note that these terms often appear even under the simple uniaxial tension due to the coupling of the elastic constants in composite laminates. In the subsequent section, we discuss the existence of the homogeneous solution $\mathbf{q}^h$ and the particular solution $\mathbf{q}^{p_0}$ and $\mathbf{q}^{p_1}$, respectively.

3. HOMOGENEOUS SOLUTIONS

For the existence of nontrivial homogeneous solutions from eqn (15), we have

$$|\mathbf{K}_c(\delta_n)| = 0, \tag{16}$$

which determines the eigenvalues δ_n. When the eigenvalues δ_n are known, within unknown constants the eigenvectors $\mathbf{q}^h$ are computed from eqn (15) by a proper normalization. The asymptotic form of homogeneous solutions for the stress and displacement is given by

$$\mathring{u}_i^h(r,\phi) = \sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n+1}[C_{kn}H_{ik}\zeta_k^{\delta_n+1} + C_{(k+3)}\bar{H}_{ik}\bar{\zeta}_k^{\delta_n+1}]/(1+\delta_n), \tag{17a}$$

$$\mathring{\sigma}_{ij}^h = \sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n}[C_{kn}G_{ijk}\zeta_k^{\delta_n} + C_{(k+3)n}\bar{G}_{ijk}\bar{\zeta}_k^{\delta_n}], \tag{17b}$$

where the expressions for H_{ik} and G_{ijk} are given in Appendix A. To take only the real part of eqns (17a,b), we may introduce

$$C_{kn} = 1/2(\gamma_{1n} - i\gamma_{2n})b_{kn} \quad \text{for complex } \delta_n, \ \mathrm{Im}\,[\delta_n] > 0, \tag{17c}$$

$$C_{kn} = 1/2\gamma_{3n}b_{kn} \quad \text{for real } \delta_n, \tag{17d}$$

where b_{kn} are eigenvectors, computed by a proper normalization, and γ_{1n}, γ_{2n} and γ_{3n} are real constants to be determined to complete the solution. It is noted that the unknown solution vectors γ_{1n}, γ_{2n} and γ_{3n}, determined from the far field condition or the remote boundary condition, will take different numerical values according to the way the eigenvectors b_{kn} are normalized, but the resulting solution C_{kn} will remain unchanged. Denoting the unknown constant γ_{1n}, γ_{2n}, γ_{3n} by β_n for simplicity, we may recast eqn (17a,b) into the following form:

$$\mathring{u}_i^h(r,\phi) = \sum_{n=1}^{\infty}\beta_n g_{in}(r,\phi;\delta_n), \quad \mathring{\sigma}_{ij}^h(r,\phi) = \sum_{n=1}^{\infty}\beta_n f_{ijn}(r,\phi;\delta_n), \tag{17e}$$

where g_{in} and f_{ijn} denote the known eigenfunctions corresponding to the n-th eigenvalue δ_n.

The power type eigenfunction expansion (7) fails to be complete when the algebraic multiplicity is greater than the geometric multiplicity, i.e. there are not sufficient sets of the power type eigenvectors associated with these multiple eigenvalues (Dempsey and Sinclair, 1979; Ting and Chou, 1981). Dempsey and Sinclair (1979) resolved this difficulty by introducing logarithmic eigenfunctions which ensure the existence of the sets of eigenvectors enough to span the solution. Subsequently this was extended to the problem of anisotropic composite laminates by Ting and Chou (1981). The existence of the logarithmic eigenfunctions can be examined by calculating the algebraic multiplicity of the eigenvalues and the rank of the associated coefficient matrices in eqn (15). For the present problem, the homogeneous solution is nothing but the solution for the generalized plane strain problems within the context of which the complete solution was reported in Ting and Chou (1981).

4. PARTICULAR SOLUTIONS

4.1. *Existence of the polynomial form of particular solutions*

To obtain the particular solutions for stress and displacement when δ_n are 0 and 1, we may choose eigenvector D_{kn} ($n = 0, 1$) as

$$D_{kn} = \tfrac{1}{2}(a_{kn} - i\hat{a}_{kn}),$$

where a_{kn} and $\hat{a}_{kn}$ are real. In the cylindrical coordinate of eqn (11), the displacement and stress components $\mathring{u}^{\mathrm{p}}_i$ and stress $\mathring{\sigma}^{\mathrm{p}}_{ij}$ in the rotated Cartesian coordinate system may be written as

$$\mathring{u}^{\mathrm{p}}_i(r,\phi,x_3) = \sum_{\delta_n=0}^{1}\sum_{k=1}^{3} r^{\delta_n-1}[a_{kn}\,\mathrm{Re}\,(H_{ik}\zeta_k^{\delta_n+1}) + \hat{a}_{kn}\,\mathrm{Im}\,(H_{ik}\zeta_k^{\delta_n+1})]/(1+\delta_n) + \mathring{u}^{\lambda}_i(r,\phi,x_3), \tag{18a}$$

$$\mathring{\sigma}^{\mathrm{p}}_{ij}(r,\phi) = \sum_{\delta_n=0}^{1}\sum_{k=1}^{3} r^{\delta_n}[a_{kn}\,\mathrm{Re}\,(G_{ijk}\zeta_k^{\delta_n}) + \hat{a}_{kn}\,\mathrm{Im}\,(G_{ijk}\zeta_k^{\delta_n})] + \mathring{\sigma}^{\lambda}_{ij}(r,\phi), \tag{18b}$$

where

$$u^{\lambda}_i(r,\phi,x_3) = \tfrac{1}{2}\lambda_i r^2\cos^2\phi + u^{\mathrm{A}}_i(r,\phi,x_3),$$

$$\sigma^{\lambda}_{ij}(r,\phi) = r\sum_{k=1}^{3} C_{ijkl}\lambda_k\cos\phi + C_{ij33}A_1$$
$$+ r[C_{ij33}\cos\phi A_2 + C_{ij33}\sin\phi A_3 + (C_{ij23}\cos\phi - C_{ij13}\sin\phi)A_4],$$

$$\mathring{u}^{\lambda}_i = u^{\lambda}_m l_{mi}, \quad \mathring{\sigma}^{\lambda}_{ij} = \sigma^{\lambda}_{mn} l_{mi} l_{nj}$$

with l_{ij} being the transformation matrix given in Appendix A.

For convenience, we set

$$\mathbf{a}^{\mathrm{p}_0} = \{a_{10}, a_{20}, a_{30}, \hat{a}_{10}, \hat{a}_{20}, \hat{a}_{30}, a'_{10}, a'_{10}, a'_{10}, \hat{a}'_{10}, \hat{a}'_{20}, \hat{a}'_{30}\}^{\mathrm{T}},$$

$$\mathbf{a}^{\mathrm{p}_1} = \{a_{11}, a_{21}, a_{31}, \hat{a}_{11}, \hat{a}_{21}, \hat{a}_{31}, a'_{11}, a'_{11}, a'_{11}, \hat{a}'_{11}, \hat{a}'_{21}, \hat{a}'_{31}\}^{\mathrm{T}}.$$

Equations (14a,b) are then replaced by

$$\mathbf{K}^*(0)\mathbf{a}^{\mathrm{p}_0} = A_1\mathbf{b}_1, \quad \mathbf{K}^*(1)\mathbf{a}^{\mathrm{p}_1} = A_2\mathbf{b}_2 + A_3\mathbf{b}_3 + A_4\mathbf{b}_4, \tag{19a,b}$$

where $\mathbf{K}^*$ is now a real valued 12×12 square matrix. We now discuss the solution of eqns (19a,b). Existence of solution $\mathbf{a}^{\mathrm{p}_0}$ and $\mathbf{a}^{\mathrm{p}_1}$ depends upon the roots of the eqn (16). When $\delta_n = 0$ and $\delta_n = 1$ are not the roots of eqn (16), the matrices $\mathbf{K}^*(0)$ and $\mathbf{K}^*(1)$ are not singular and the unique solutions for $\mathbf{a}^{\mathrm{p}_0}$ and $\mathbf{a}^{\mathrm{p}_1}$ are assured. On the other hand, if $\delta_n = 0$ and $\delta_n = 1$ are the roots of the eqn (16), i.e. the eigenvalues for the homogeneous solution, a solution of $\mathbf{a}^{\mathrm{p}_0}$ and $\mathbf{a}^{\mathrm{p}_1}$ in eqns (19a,b) exists if and only if

$$\mathbf{s}_0\cdot\mathbf{b}_1 = 0,\quad \mathbf{s}_1\cdot(A_2\mathbf{b}_2+A_3\mathbf{b}_3+A_4\mathbf{b}_4) = 0, \tag{20a,b}$$

where $\mathbf{s}_0$ and $\mathbf{s}_1$ are eigenvectors of $\mathbf{K}^{*\mathrm{T}}(0)$ and $\mathbf{K}^{*\mathrm{T}}(1)$, respectively,

$$\mathbf{K}^{*\mathrm{T}}(0)\cdot\mathbf{s}_0 = 0,\quad \mathbf{K}^{*\mathrm{T}}(1)\cdot\mathbf{s}_1 = 0.$$

If the consistency conditions (20a,b) hold when $\delta_n = 0$ and $\delta_n = 1$ are the roots of eqn (16), the solution vectors $\mathbf{a}^{\mathrm{p}_0}$ and $\mathbf{a}^{\mathrm{p}_1}$ may include the homogeneous solution part in addition to the terms associated with the deformation parameters A_i. However, this homogeneous solution part may be neglected since it has been included in the homogeneous solution. Substitution of $\mathbf{a}^{\mathrm{p}_0}$ and $\mathbf{a}^{\mathrm{p}_1}$ into eqns (18a,b) with $\delta_n = 0$ and $\delta_n = 1$, respectively, we can then obtain the following polynomial form of particular solutions

$$\mathring{u}_i^{\mathrm{p}} = \mathring{u}_i^{\mathrm{p}_0}+\mathring{u}_i^{\mathrm{p}_1},\quad \mathring{\sigma}_{ij}^{\mathrm{p}} = \mathring{\sigma}_{ij}^{\mathrm{p}_0}+\mathring{\sigma}_{ij}^{\mathrm{p}_1}, \tag{21a,b}$$

where

$$\mathring{u}_i^{\mathrm{p}_0} = \sum_{k=1}^{3}[a_{k0}\,\mathrm{Re}\,(H_{ik}z_k)+\hat{a}_{k0}\,\mathrm{Im}\,(H_{ik}z_k)]+\delta_{i3}A_1x_3,$$

$$\mathring{\sigma}_{ij}^{\mathrm{p}_0} = \sum_{k=1}^{3}[a_{k0}\,\mathrm{Re}\,(G_{ijk})+\hat{a}_{k0}\,\mathrm{Im}\,(G_{ijk})]+C_{mn33}A_1l_{mi}l_{nj},$$

$$\mathring{u}_i^{\mathrm{p}_1} = \sum_{k=1}^{3}[a_{k1}\,\mathrm{Re}\,(H_{ik}z_k^2)+\hat{a}_{k1}\,\mathrm{Im}\,(H_{ik}z_k^2)]/2+\mathring{u}_i^{\lambda}(r,\phi,x_3)-\delta_{i3}A_1x_3,$$

$$\mathring{\sigma}_{ij}^{\mathrm{p}_1} = \sum_{k=1}^{3}[a_{k1}\,\mathrm{Re}\,(G_{ijk}z_k)+\hat{a}_{k1}\,\mathrm{Im}\,(G_{ijk}z_k)]+\mathring{\sigma}_{ij}^{\lambda}(r,\phi)-C_{mn33}A_1l_{mi}l_{nj}.$$

So far we used the Stroh formalism to determine the structure of the complete solution and to investigate the existence of the polynomial type particular solutions. From eqns (21a,b), it turns out that the polynomial type particular solutions for displacement and stress take quadratic and linear form, respectively. However, as discussed above, these solutions are valid only when $\delta_n = 0$ and $\delta_n = 1$ are not the roots of the eigenvalue eqn (16) for the homogeneous solution, or when the consistency conditions (20a,b) are fulfilled if $\delta_n = 0, 1$ are among the eigenvalues for the homogeneous solution.

4.2. *Logarithmic particular solutions*

If the consistency conditions (20a,b) do not hold when $\delta_n = 0$ and 1 are eigenvalues for the homogeneous solution, the particular solutions of polynomial form for displacement and stress due to deformation parameters A_1, A_2, A_3, and A_4 do not exist. In such a case, instead of using eqns (18a,b) we use the following logarithmic eigenfunction, as in Zwiers *et al.* (1982),

$$\mathring{u}_i^{\mathrm{p}}(r,\phi,x_3) = \sum_{\delta_n=0}^{1} r^{\delta_n+1}\left(\ln r+\frac{\partial}{\partial\delta_n}\right)\left\{\sum_{k=1}^{3}[a_{kn}\,\mathrm{Re}\,(H_{ik}\zeta_k^{\delta_n+1})+\hat{a}_{kn}\,\mathrm{Im}\,(H_{ik}\zeta_k^{\delta_n+1})]/(1+\delta_n)\right\} + \mathring{u}_i^{\lambda}(r,\phi,x_3), \tag{22a}$$

$$\mathring{\sigma}_{ij}^{\mathrm{p}}(r,\phi) = \sum_{\delta_n=0}^{1} r^{\delta_n}\left(\ln r+\frac{\partial}{\partial\delta_n}\right)\left\{\sum_{k=1}^{3}[a_{kn}\,\mathrm{Re}\,(G_{ijk}\zeta_k^{\delta_n})+\hat{a}_{kn}\,\mathrm{Im}\,(G_{ijk}\zeta_k^{\delta_n})]\right\}+\mathring{\sigma}_{ij}^{\lambda}(r,\phi). \tag{22b}$$

Employing these expressions for the particular solutions and subsequently applying the near field conditions, we can obtain the following linear equations

$$\mathbf{K}^*(0)\mathbf{a}^{p_0}(0) = 0, \quad \left\{\frac{\partial}{\partial \delta_n}[\mathbf{K}^*(\delta_n)\mathbf{a}^{p_0}(\delta_n)]\right\}_{\delta_n=0} = A_1\mathbf{b}_1, \tag{23a,b}$$

$$\mathbf{K}^*(1)\mathbf{a}^{p_1}(1) = 0, \quad \left\{\frac{\partial}{\partial \delta_n}[\mathbf{K}^*(\delta_n)\mathbf{a}^{p_1}(\delta_n)]\right\}_{\delta_n=1} = A_2\mathbf{b}_2 + A_3\mathbf{b}_3 + A_4\mathbf{b}_4. \tag{24a,b}$$

The solution $\mathbf{a}^{p_0}$ for eqns (23a,b), in conjunction with eqns (22a,b), will give the particular solutions for the deformation parameter A_1, which represents the uniform extension, and it was discussed in Zwiers *et al.* (1982). We therefore focus upon the solution $\mathbf{a}^{p_1}$ for eqns (24a,b), which constitutes the particular solutions for the deformation parameters A_2, A_3 and A_4, which are related to the curvature about the x_2-axis and the x_1-axis, and the twist along the x_3-axis, respectively. For simplicity, we write eqns (24a,b) as

$$\mathbf{K}^*\mathbf{a}^{p_1} = 0, \quad \left(\frac{\partial \mathbf{K}^*}{\partial \delta_n}\right)\mathbf{a}^{p_1} + \mathbf{K}^*\left(\frac{\partial \mathbf{a}^{p_1}}{\partial \delta_n}\right) = A_2\mathbf{b}_2 + A_3\mathbf{b}_3 + A_4\mathbf{b}_4, \tag{25a,b}$$

where all quantities on the left hand side of eqns (25a,b) are evaluated at $\delta_n = 1$. Equations (25a,b) comprise a system of 24 equations for $\mathbf{a}^{p_1}$ and $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$, whose elements are a_{k1}, $\hat{a}_{k1}$, a'_{k1}, $\hat{a}'_{k1}$ and $\partial a_{k1}/\partial \delta_n$, $\partial \hat{a}_{k1}/\partial \delta_n$, $\partial a'_{k1}/\partial \delta_n$, $\partial \hat{a}'_{k1}/\partial \delta_n$ $(k = 1\text{–}3)$.

The system of eqns (25a,b) has a unique solution for $\mathbf{a}^{p_1}$ if (Dempsey and Sinclair, 1979)

$$\partial^N|\mathbf{K}^*|/\partial \delta_n^N \neq 0, \quad N = n - m, \tag{26}$$

where n and m are, respectively, the order and the rank of $\mathbf{K}^*$. However, it is rather difficult to prove eqn (26) analytically or numerically. Instead, we may regard eqns (25a,b) as a system of 24 equations for $\mathbf{a}^{p_1}$ and $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$, and solve the system numerically. In general, $\mathbf{a}^{p_1}$ is unique while $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$ has a particular solution and at least one arbitrary solution. The fact that $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$ has at least one arbitrary solution is obvious from eqn (25b) because the coefficient of $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$ is $\mathbf{K}^*$, which has singularity of at least order one. However, one arbitrary solution for $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$ is equivalent to the homogeneous solution for $\delta_n = 1$, and this term may be left out in the particular solution once included in the homogeneous solution (17a,b). Substitution of $\mathbf{a}^{p_1}$ and $(\partial \mathbf{a}^{p_1}/\partial \delta_n)$ into eqns (22a,b) with $\delta_n = 1$ provides the particular solutions for displacement and stress, including logarithmic terms:

$$\mathring{u}_i^{p_1} = \sum_{k=1}^{3} \{a_{k1} \operatorname{Re}[H_{ik}z_k^2(2\ln z_k - 1)] + \hat{a}_{k1} \operatorname{Im}[H_{ik}z_k^2(2\ln z_k - 1)]\}/4$$
$$+ \sum_{k=1}^{3} \left[\frac{\partial a_{k1}}{\partial \delta_n} \operatorname{Re}(H_{ik}z_k^2) + \frac{\partial \hat{a}_{k1}}{\partial \delta_n} \operatorname{Im}(H_{ik}z_k^2)\right] \Big/ 2 + \mathring{u}_i^{\lambda}(r, \phi, x_3) - \delta_{i3}A_1x_3,$$

$$\mathring{\sigma}_{ij}^{p_1} = \sum_{k=1}^{3} [a_{k1} \operatorname{Re}(G_{ijk}z_k \ln z_k) + \hat{a}_{k1} \operatorname{Im}(G_{ijk}z_k \ln z_k)]$$
$$+ \sum_{k=1}^{3} \left[\frac{\partial a_{k1}}{\partial \delta_n} \operatorname{Re}(G_{ijk}z_k) + \frac{\partial \hat{a}_{k1}}{\partial \delta_n} \operatorname{Im}(G_{ijk}z_k)\right] + \mathring{\sigma}_{ij}^{\lambda}(r, \phi) - C_{mn33}A_1l_{mi}l_{nj}.$$

The particular solutions $\mathring{u}_i^{p_0}$ and $\mathring{\sigma}_{ij}^{p_0}$, constructed from $\mathbf{a}^{p_0}$ and $(\partial \mathbf{a}^{p_0}/\partial \delta_n)$ and related to the deformation parameter A_1, will include the terms like $z_k \ln z_k$ and $\ln z_k$, respectively (Zwiers *et al.*, 1982). Taken together, we can obtain the particular solutions (22a,b) as

$$\mathring{u}_i^{\mathrm{p}} = \mathring{u}_i^{\mathrm{p}_0} + \mathring{u}_i^{\mathrm{p}_1}, \quad \mathring{\sigma}_{ij}^{\mathrm{p}} = \mathring{\sigma}_{ij}^{\mathrm{p}_0} + \mathring{\sigma}_{ij}^{\mathrm{p}_1}. \tag{27a,b}$$

From the aforementioned discussion, ahead of the wedge tip the asymptotic stress includes $\ln r$ and $r \ln r$ terms as r goes to zero while the asymptotic displacement $r \ln r$ and $r^2 \ln r$ terms. Although the solutions obtained here are for composite laminates for which the polynomial type particular solutions do not exist, application of the logarithmic type particular solutions to composite laminates for which the polynomial type particular solutions exist yields $\mathbf{a}^{\mathrm{p}_0} = 0$ and $\mathbf{a}^{\mathrm{p}_1} = 0$ in eqns (23a,b) and (24a,b), respectively.

5. NUMERICAL EXAMPLES AND DISCUSSION

5.1. *Numerical results for particular solutions*

In this section, we choose three examples, the free edge problem, and the opened and closed delamination crack problems, to examine particular solutions, and to check whether the polynomial type particular solutions exist or not.

For the purpose of illustrating a particular solution, we consider a case wherein each of the deformation parameters A_1, A_2, A_3 and A_4 is applied alone, respectively. A solution for generic loadings, such as uniaxial tension, pure bending, torsion, or a combination of these loading modes, may be obtained from a linear combination of the solutions for these four cases (when there are no configuration change like the contact of crack faces) with the aid of the end conditions (Kim and Im, 1994): these end conditions represent the force resultants and the moment resultants in terms of stress, that is, implicitly in terms of A_i ($i = 1, 2, 3, 4$). The special cases of $A_2\mathbf{b}_2 + A_3\mathbf{b}_3 + A_4\mathbf{b}_4 = 0$ are excluded here. In each case, eqns (20a,b) are then rewritten as follows,

$$\mathbf{s}_0 \cdot \mathbf{b}_1 = 0, \quad \mathbf{s}_1 \cdot \mathbf{b}_2 = 0, \quad \mathbf{s}_1 \cdot \mathbf{b}_3 = 0, \quad \mathbf{s}_1 \cdot \mathbf{b}_4 = 0, \tag{28a,b,c,d}$$

where $\mathbf{s}_0$ and $\mathbf{s}_1$ are eigenvectors of $\mathbf{K}^{*\mathrm{T}}(0)$ and $\mathbf{K}^{*\mathrm{T}}(1)$

$$\mathbf{K}^{*\mathrm{T}}(0)\mathbf{s}_0 = 0, \quad \mathbf{K}^{*\mathrm{T}}(1)\mathbf{s}_1 = 0.$$

For numerical computation, we employ the following material data for the graphite epoxy T300/5208 (Whitcomb, 1989),

$$E_L = 134\ \mathrm{GPa}, \quad E_T = E_Z = 10.2\ \mathrm{GPa},$$
$$G_{LT} = G_{LZ} = 5.52\ \mathrm{GPa}, \quad G_{TZ} = 3.43\ \mathrm{GPa},$$
$$\nu_{LT} = \nu_{LZ} = 0.3, \quad \nu_{TZ} = 0.49,$$

where L, T and Z indicate principal material axes along fiber, transverse and thickness directions, respectively. From this data, the eigenvalues μ_k (k = 1–6) and the associated eigenvectors v_{ik} are obtained. The values of λ_k are then obtained from eqn (10) and τ_{ik}, H_{ik} and G_{ijk} are calculated from the expressions in Appendix A. From these results, we can finally calculate $\mathbf{K}^*(0)$, $\mathbf{K}^*(1)$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$. Ideally speaking, fiber reinforced laminated composites must be transversely isotropic with the axis of symmetry lined up along the fiber direction. However, the material data above, which has been determined experimentally, slightly deviates from the transversely isotropic case, so that no double roots appear in the eigenvalue μ_k.

From the computational results, it turns out that for all of the ply orientations, the integers $\delta_n = 0$ and $\delta_n = 1$ are both roots of the eigenvalue eqn (16) for the homogeneous solution. Moreover, there are three eigenvectors of $\mathbf{K}^{*\mathrm{T}}(0)\mathbf{s}_0 = 0$ and $\mathbf{K}^{*\mathrm{T}}(1)\mathbf{s}_1 = 0$ for the opened delamination crack problem, and similarly the four eigenvectors of each matrix for the closed crack, while there are two eigenvectors of $\mathbf{K}^{*\mathrm{T}}(0)\mathbf{s}_0 = 0$ and one eigenvector of $\mathbf{K}^{*\mathrm{T}}(1)\mathbf{s}_1 = 0$ for free edge problem. For each of these three cases, we now check whether

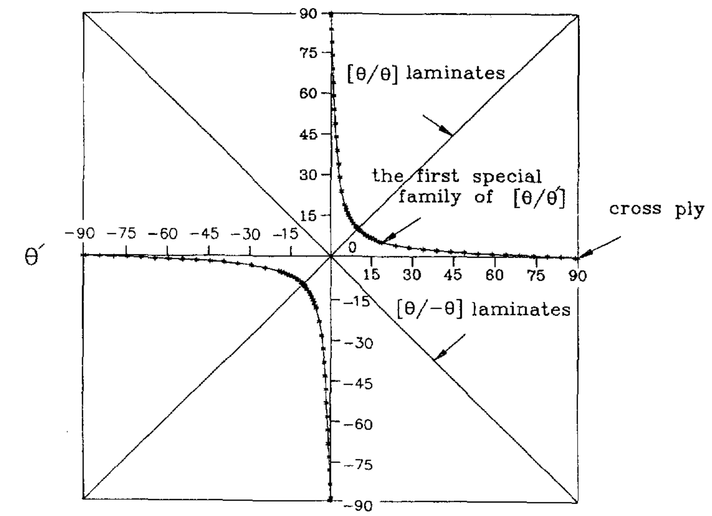

Fig. 2. Existence of the polynomial type particular solutions for free edge problem under uniform extension (A_1).

the polynomial type particular solutions exist or not. For the opened and closed delamination crack problem, the consistency conditions (28a–d) are all satisfied by every set of eigenvectors regardless of any combination of ply orientations. This means that the polynomial type particular solutions exist for all of the fiber angles under typical deformations such as extension, bending and torsion. On the other hand, for a free edge problem: (i) eqn (28a) holds for $[\theta/\theta]$ laminates which are of a single homogeneous ply, the cross ply, the angle ply of $[\theta/-\theta]$ and the first special family of $[\theta/\theta']$ laminates, which are represented by the line starting at [0/90] and ending at [90/0] in Fig. 2; (ii) eqn (28b) holds for all of the ply orientations; (iii) eqn (28c) holds for $[\theta/\theta]$ laminates, the cross ply, the type of angle ply $[\theta/-\theta]$ and the second special family of $[\theta/\theta']$ laminates (see Fig. 3); and (iv) eqn (28d) holds for $[\theta/\theta]$ laminates only. These results imply that when the deformation parameter A_1, related to the extension along the x_3-axis is imposed, the polynomial type particular solutions exist for $[\theta/\theta]$ laminates, the cross ply, the angle ply of $[\theta/-\theta]$ and the first special family of $[\theta/\theta']$, while for A_2, the polynomial type particular solutions exist for all combination of the ply orientations. For the deformation parameter A_3, the polynomial

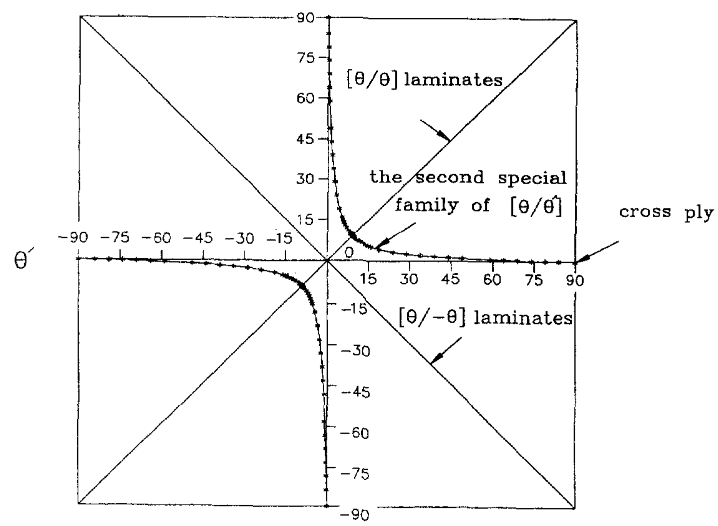

Fig. 3. Existence of the polynomial type particular solution for free edge problem under curvature in the x_2–x_3 plane (A_3).

type particular solutions exist for $[\theta/\theta]$ laminates, the cross ply, the angle ply of $[\theta/-\theta]$ and the second special family of $[\theta/\theta']$. However, when the deformation parameter A_4, related to twist along the x_3-axis, is applied, the polynomial type particular solutions hold for $[\theta/\theta]$ laminates only. These are summarized in Table 1.

5.2. *The nature of the singular stress field*

Now, we examine the nature of the singular stress field at the tip of a wedge, including the structure of solution. Particularly we investigate the linkage between the near field mode mixity and the far field mode mixity, which implies the relation between the two ratios of shear stress to normal stress at the wedge tip and at the remote boundary. It will be illustrated how the elastic symmetry, the multiplicity of the singular eigenvalues δ^s and the existence of the imaginary part in δ^s are linked to the nature of the singular stress field including the mode mixity. To discuss the mode mixity we define the two phase angles Φ_1^* and Φ_2^* for a singular asymptotic field resulting from a real singularity δ_r^s ahead of the wedge tip as

$$\Phi_1^*(\delta_r^s) = \lim_{r\to 0} \tan^{-1}\{\sigma_{12}(r,0;\delta_r^s)/\sigma_{22}(r,0;\delta_r^s)\}, \tag{29a}$$

$$\Phi_2^*(\delta_r^s) = \lim_{r\to 0} \tan^{-1}\{\sigma_{23}(r,0;\delta_r^s)/\sigma_{22}(r,0;\delta_r^s)\} \tag{29b}$$

where $\sigma_{12}(r,0;\delta_r^s)$ indicates the singular asymptotic stress field associated with δ_r^s. Further, let Φ_1^∞ and Φ_2^∞ denote the corresponding phase angles of the applied loading on the remote boundary.

Figures 4–6 show the singular eigenvalues for various wedge angles $[\alpha/\alpha]$ in laminate strips of a single homogeneous ply or $[\theta/\theta]$ laminates. In general, there are three discrete real singular eigenvalues, which we name the first, the second and the third eigenvalue, respectively. Note that for a real eigenvalue with no multiplicity, there exists one scaling parameter K_r characterizing the corresponding singular stress field when eigenvectors b_{ks} have been properly normalized,

$$\sigma_{ij}^s(r,0;\delta_r^s) = \frac{K_r r^{\delta_r^s}}{\sqrt{2\pi}} \mathrm{Re}\left[\sum_{k=1}^{3} b_{ks}\tau_{ijk}\right], \quad K_r = \sqrt{2\pi}\gamma_{3s}. \tag{30}$$

In this expression, only the scaling parameter K_r is dependent upon the far field loading at the remote boundary. That is, for a real singular eigenvalue δ_r^s the loading at the remote boundary effects the asymptotic stress field associated with δ_r^s only through the scaling parameter K_r. As a consequence the near field mode mixities $\Phi_1^*(\delta_r^s)$ and $\Phi_2^*(\delta_r^s)$ will be

Table 1. Existence of the polynomial type particular solution under generic deformations

Two types of wedge problem	Delamination crack (either opened or closed)				Free edge			
	Loading parameters							
Ply orientation	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
$[\theta/\theta]$ Laminates	0	0	0	0	0	0	0	0
Cross ply laminates	0	0	0	0	0	0	0	×
$[\theta/-\theta]$ Laminates	0	0	0	0	0	0	0	×
The first special family	0	0	0	0	0	0	×	×
The second special family	0	0	0	0	×	0	0	×
$[\theta/\theta']$ Laminates	0	0	0	0	×	0	×	×

(1) $[\theta/\theta']$ laminates mean arbitrary ply orientations except for $[\theta/\theta]$ laminates, the cross ply, the angle ply of $[\theta/-\theta]$ and two special families of $[\theta/\theta']$ laminates described in Section 5.
(2) "0" means the polynomial type solutions exist.
(3) " × " means that the logarithmic type solutions exist.
(4) The two special families of $[\theta/\theta']$ laminates are depicted in Figs 2 and 3, respectively.

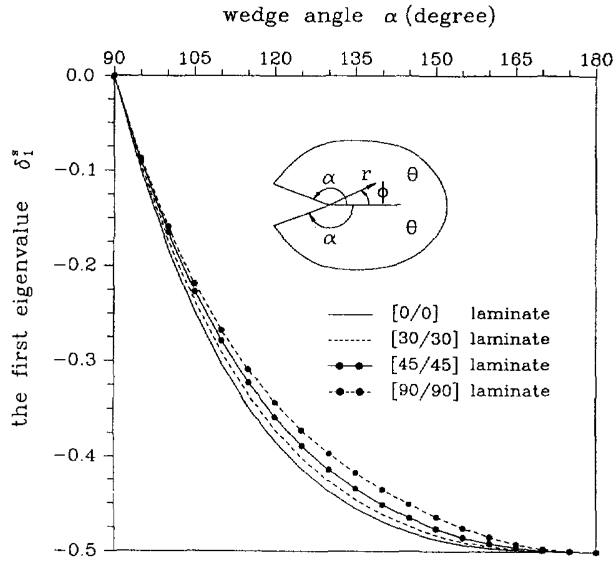

Fig. 4. The first singular eigenvalues for various wedge angles in laminates of a homogeneous ply.

independent of the far field mode mixities Φ_1^∞ and Φ_2^∞ if K_r is non-zero (when δ_r^s is a single root).

For $\theta = 0$ and 90, the laminate becomes exactly orthotropic, so that the modes I, II, III, which are related to the stress components σ_{22}, σ_{12} and σ_{23} ahead of the tip in the wedge, respectively, are decoupled from one another. The first eigenvalue δ_1^s is associated with the mode I, the second eigenvalue δ_2^s with the mode II and the third eigenvalue δ_3^s with the mode III. For each of the three modes, which have different singularities, there exists a single real scaling parameter characterizing the near tip field associated with the respective singularity. The phase angle Φ_1^* will be 0 and $\pi/2$ for the first singularity δ_1^s and the second singularity δ_2^s, associated with in-plane deformations, respectively, and the phase angle Φ_2^* will be $\pi/2$ for the third singularity δ_3^s, associated with anti-plane deformations. Which of the three discrete singularities appears depends upon the remote loading. As shown in Fig. 5, the second singular eigenvalue δ_2^s for $\theta = 0$ disappears at $\alpha = 128.3°$. This threshold wedge angle is slightly different from $\alpha = 128.7°$, which is for the case of isotropic

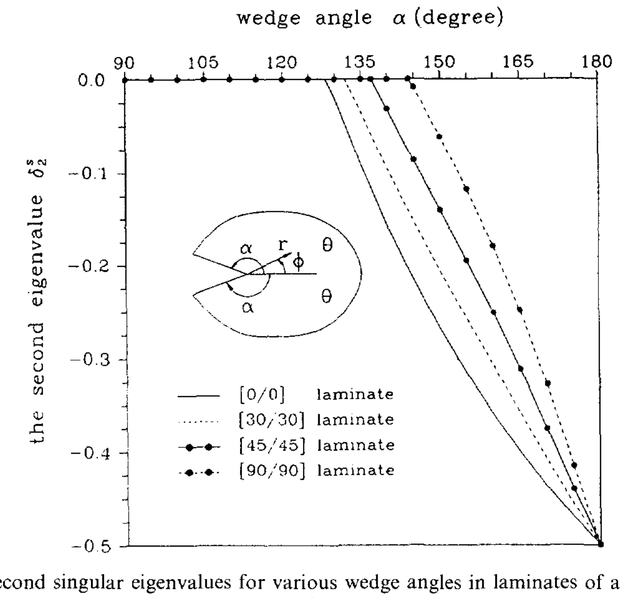

Fig. 5. The second singular eigenvalues for various wedge angles in laminates of a homogeneous ply.

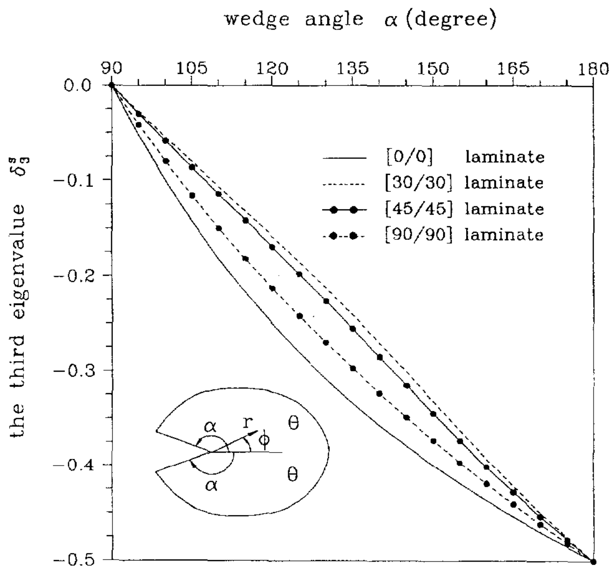

Fig. 6. The third singular eigenvalues for various wedge angles in laminates of a homogeneous ply.

or transversely isotropic materials (Zhao and Hahn, 1992), because the material considered here is not quite transversely isotropic as discussed in Section 5.1.

For the ply orientations other than $\theta = 0$ and 90, the laminates behave like a monoclinic material with the x_1–x_3 plane being the plane of reflection symmetry, so that the deformations involving the stress components σ_{12} and σ_{23} are decoupled from the deformations involving the rest of the stress components. The numerical results show that the first singular eigenvalue δ_1^s is associated with the mode I, and the second and the third eigenvalues δ_2^s, δ_3^s are related to the mode II and III, which are coupled with each other in general. Similarly, the phase angles Φ_1^* and Φ_2^* will be both zero for the first singularity, and both $\pi/2$ for δ_2^s and δ_3^s. Regardless of the ply orientation θ and the wedge angle α, $|\delta_1^s| > |\delta_3^s| > |\delta_2^s|$.

The singular eigenvalues and the phase angles for various wedge angles $[\alpha/\alpha]$ in a laminated composite strip of different adjacent plies $[\theta/\theta']$ are shown for $[60/-45]$ and $[90/0]$ laminates in Figs 7 and 8. The results for the $[60/-45]$ laminate were found to be typical of those for the angle ply laminates $[\theta/\theta']$. As in the laminated strips of a single homogeneous ply, there exist three eigenvalues δ_1^s, δ_2^s and δ_3^s for most of the wedge angles, however, the second singular eigenvalue δ_2^s disappears around $\alpha = 125°$–$130°$ depending upon the ply orientations $[\theta/\theta']$, and the third singular eigenvalue δ_3^s disappears at $\alpha = 90°$ regardless of combinations of ply orientations $[\theta/\theta']$ (Figs 7 and 8). Hence only one singularity δ_1^s appears for the free edge problem. The eigenvalues δ^s become complex for large wedge angles α, i.e. when a wedge approaches a crack, as shown in Figs 7 and 8. As in the case of single homogeneous ply, the inequality $|\delta_1^s| > |\delta_3^s| > |\delta_2^s|$ still holds for most of the wedge angles except for large values of α near 180° wherein two of the three singular eigenvalues become complex. For each of the real eigenvalues (with no multiplicity) in laminated composite strips of different adjacent plies, as in the case of a single homogeneous ply, there exists one real scaling parameter K_r characterizing the associated singular stress field for a given normalization of the eigenvectors b_{ks}. The asymptotic expression for the near tip stress field is given as eqn (30) for the upper ply, and for the lower ply it will take the same form with the primed quantities but with the same K_r as for the upper ply. As a consequence, the near field mode mixities Φ_1^* and Φ_2^* associated with a real singular eigenvalue will be independent of the far field mode mixities, once the associated eigenmode is activated by the far field loading, which is probable under a generic loading due to the coupling of elastic constants particularly for angle ply laminates. In Figs 7 and 8, $\Phi_1^*(\delta_1^s)$ and $\Phi_2^*(\delta_1^s)$, associated with the real dominant singularity δ_1^s, are plotted for illustration. Both of Φ_1^* and Φ_2^* increase abruptly just before the eigenvalue δ_1^s becomes complex as the wedge angle α approaches 180°. This indicates that the shear stress components σ_{12} and σ_{23}

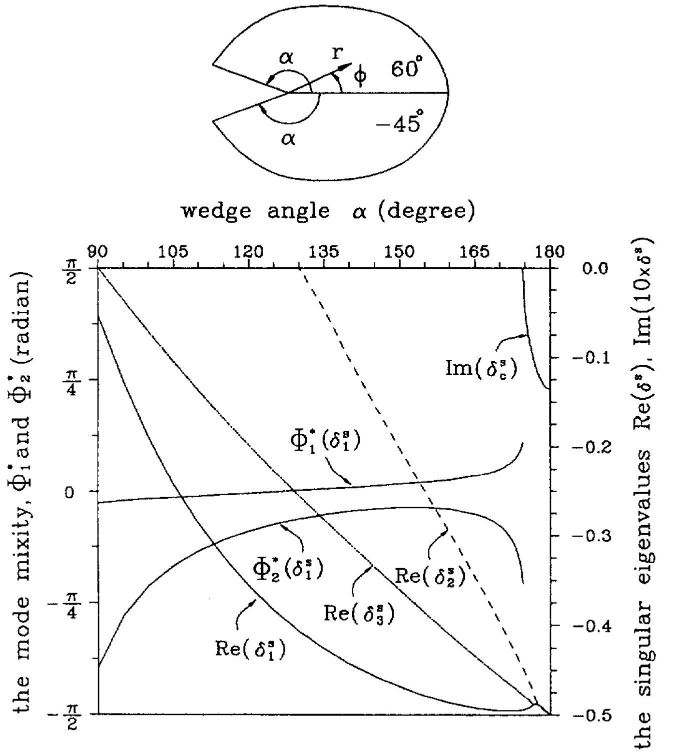

Fig. 7. The singular eigenvalues and mode mixities for various wedge angles in the [60/−45] composite laminate.

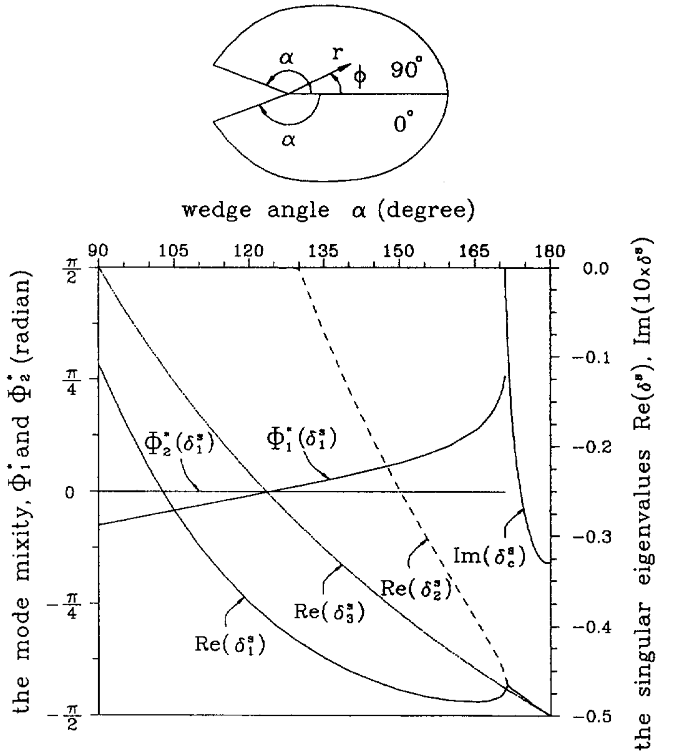

Fig. 8. The singular eigenvalues and mode mixities for various wedge angles in the [90/0] composite laminate.

become relatively important in the near tip field as a wedge approaches a crack, and therefore this suggests that the shear failure may be relatively important as a wedge approaches a crack.

Because of the coupling between the elastic constants, resulting from distortion on the interface, the mode decomposition does not occur in a laminated composite strip of two different adjacent plies, except for the special case of an orthotropic material [90/0]. For the [90/0] laminate, the deformations associated with the stress components σ_{11}, σ_{22}, σ_{33} and σ_{12} are decoupled from deformations related to σ_{23} and σ_{31}. The first and the second eigenvalue δ_1^s and $\delta_2^s$are associated with the stress components σ_{11}, σ_{22}, σ_{33}, σ_{12}, and the third eigenvalue δ_3^s with the stress components σ_{23} and σ_{31}. Note that this is consistent with the case for a laminated strip of a single homogeneous ply [90/90] or [0/0].

For complex singular eigenvalues δ^s, which always appear as a pair of complex conjugates, there exists a complex scaling parameter characterizing a wedge tip field for a given normalization of the eigenvectors b_{ks} as in an interfacial crack of anisotropic materials (see Suo, 1990). Let δ_c^s and K_c denote the complex singular eigenvalue and the complex scaling parameter, respectively. Then the corresponding singular asymptotic stress field may be written as

$$\sigma_{ij}^s(r,0;\delta_c^s) = \frac{r^{\mathrm{Re}(\delta_c^s)}}{\sqrt{2\pi}}[\mathrm{Re}\,(K_c r^{i\eta})\tilde{\sigma}_{ij}^{\mathrm{I}} + \mathrm{Im}\,(K_c r^{i\eta})\tilde{\sigma}_{ij}^{\mathrm{II}}], \tag{31}$$

where

$$\eta = \mathrm{Im}\,(\delta_s),\quad K_c = \sqrt{2\pi}(\gamma_{1s} - i\gamma_{2s}),\quad \tilde{\sigma}_{ij}^{\mathrm{I}} = \mathrm{Re}\left[\sum_{k=1}^{3}(b_{ks}\tau_{ijk} + b_{(k+3)s}\bar{\tau}_{ijk})\right],$$

$$\tilde{\sigma}_{ij}^{\mathrm{II}} = -\mathrm{Im}\left[\sum_{k=1}^{3}(b_{ks}\tau_{ijk} + b_{(k+3)s}\bar{\tau}_{ijk})\right]. \tag{32}$$

The complex scaling parameter K_c, which affects the near field mode mixities now, is dependent upon the applied loading as well as the geometry and the material properties. Consequently the mode mixities $\Phi_1^*(\delta_c^s)$ and $\Phi_2^*(\delta_c^s)$ are dependent upon the mode mixities of the applied loading as well as the geometry and the material properties of the two adjacent plies.

For a wedge with α close to 180°, the two eigenvalues appear as a conjugate pair of complex numbers and the remaining eigenvalue as a real number. The near tip stress field may be written as a sum of these two singular terms:

$$\sigma_{ij}^s(r,0) = \frac{r^{\mathrm{Re}(\delta_c^s)}}{\sqrt{2\pi}}\left[\mathrm{Re}\,(K_c r^{i\eta})\tilde{\sigma}_{ij}^{\mathrm{I}} + \mathrm{Im}\,(K_c r^{i\eta})\tilde{\sigma}_{ij}^{\mathrm{II}}\right] + \frac{K_r r^{\delta_r^s}}{\sqrt{2\pi}}\mathrm{Re}\left[\sum_{k=1}^{3} b_{ks}\tau_{ijk}\right]. \tag{33}$$

For $\alpha = 180°$, this will reduce to the asymptotic form for an opened interfacial crack in an anisotropic material, which was reported by Suo (1990).

For a closed interfacial crack ($\alpha = 180°$) in the absence of friction, only real singularity $\delta_r^s = -1/2$ appears as a double root, and there exist two real parameters $K_r^{(1)}$, $K_r^{(2)}$ characterizing the near tip singular stress field:

$$\sigma_{ij}^s(r,0) = r^{-1/2}\left\{\frac{K_r^{(1)}}{\sqrt{2\pi}}\mathrm{Re}\left[\sum_{k=1}^{3} b_{ks}^{(1)}\tau_{ijk}\right] + \frac{K_r^{(2)}}{\sqrt{2\pi}}\mathrm{Re}\left[\sum_{k=1}^{3} b_{ks}^{(2)}\tau_{ijk}\right]\right\}, \tag{34}$$

where

$$K_r^{(1)} = \sqrt{2\pi}\gamma_{3s}^{(1)}, \quad K_r^{(2)} = \sqrt{2\pi}\gamma_{3s}^{(2)}.$$

The relative magnitude of the two real parameters $K_r^{(1)}$, $K_2^{(2)}$ will depend upon the applied loadings at the remote boundary. This is the consequence of the multiplicity (double roots) in the eigenvalue $\delta_r^s = -1/2$.

For the free edge problem, for which only one stress singularity occurs, the existence of one single scalar scaling parameter K_r characterizing the near tip stress field around a wedge is of paramount importance in relation to fracture or failure initiation at a free edge. To characterize free edge stress field, Wang and Choi (1982) introduced the following six "boundary layer stress intensity factors" K_{ij} defined by

$$K_{ij} = \lim_{r \to 0} r^{-\delta_r^s} \sigma_{ij}^s(r, 0; \delta_r^s). \quad (35)$$

However, the aforementioned discussion shows that only one single real scalar parameter K_r characterizes the near tip stress field around a free edge, and therefore just this single parameter controls the near field behavior such as fracture or failure initiation at a free edge regardless of the geometry and the loading mode at the far field, i.e. fracture or failure initiation will occur for a free edge composed of a given combination of two adjacent materials only if the parameter K_r reaches a certain critical value regardless of the far field geometry and the loading modes.

6. CONCLUSION

Based upon the Stroh formalism and Lekhnitskii's representation, the structure of the asymptotic solution has been examined for the boundary layer on the wedge type cross section of a laminated composite strip under the generalized plane deformations. A basis for obtaining the complete numerical solutions to the wedge problems, including free edge or delamination problems in laminated composite strips under the aforementioned generic loadings has been established (Kim and Im, 1994).

Numerical results for the case of the graphite epoxy T300/5208 show the following:

(1) For the opened and the closed delamination cracks: a set of consistency conditions hold for all of the ply orientations, and the logarithmic terms do not appear in the particular solution for any of the deformations—extension, curvature and twist.

(2) For the free edge, there may exist either the polynomial solutions or the logarithmic solutions depending upon the combination of the ply orientations and upon the type of deformations.

(3) There exists a single real scaling parameter for the asymptotic eigenmode associated with a real singular eigenvalue with no multiplicity, and consequently the near field mode mixity of a singular eigenmode for a real eigenvalue becomes independent of the far field mode mixity once the eigenmode is activated.

(4) For the dominant singularity, the interfacial shear stress components increase as a wedge approaches a crack, which suggests that the shear failure may become relatively important for large wedge angles.

(5) For a free edge problem, wherein only one real singular eigenvalue exists, one single real scaling parameter governs the near field response, i.e. fracture or failure initiation at a free edge regardless of details of geometry and loading at the far field. For a wedge approaching a crack, wherein a conjugate pair of complex singularities and one real singularity exist, each of one complex parameter and one real parameter will characterize the asymptotic stress field associated with the corresponding complex and real eigenvalue, respectively.

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APPENDIX A

Expressions for $\mathring{U}_i^{\mathrm{h}}$, $\mathring{U}_i^{\mathrm{p}}$, $\mathring{u}_i^{\mathrm{A}}$, $\mathring{\sigma}_{ij}^{\mathrm{h}}$, $\mathring{\sigma}_{ij}^{\mathrm{p}}$ are given as

$$\mathring{U}_i^{\mathrm{h}}(r,\phi) = \sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n-1}[C_{kn}H_{ik}\zeta_k^{\delta_n+1} + C_{(k-3)}\bar{H}_{ik}\bar{\zeta}_k^{\delta_n+1}]/(1+\delta_n),$$

$$\mathring{U}_i^{\mathrm{p}}(r,\phi) = r\sum_{k=1}^{3}[D_{k0}H_{ik}\zeta_k + \bar{D}_{k0}\bar{H}_{ik}\bar{\zeta}_k] + r^2\sum_{k=1}^{3}\tfrac{1}{2}[D_{k1}H_{ik}\zeta_k^2 + \bar{D}_{k1}\bar{H}_{ik}\bar{\zeta}_k^2] + \tfrac{1}{2}\lambda_m(r\cos\phi)^2 l_{mi},$$

$$u_i^{\mathrm{A}}(r,\phi,x_3) = \delta_{i1}(-A_2x_3^2/2 - A_4\sin\phi x_3)\delta_{i2}(-A_3x_3^2/2 + A_4 r\cos\phi\, x_3) + \delta_{i3}(A_2 r\cos\phi + A_3 r\sin\phi + A_1)x_3,$$

$$\mathring{u}_i^{\mathrm{A}} = u_m^{\mathrm{A}} l_{mi},$$

$$\mathring{\sigma}_{ij}^{\mathrm{h}}(r,\phi) = \sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n}[C_{kn}G_{ijk}\zeta_k^{\delta_n} + C_{(k+3)n}\bar{G}_{ijk}\bar{\zeta}_k^{\delta_n}],$$

$$\mathring{\sigma}_{ij}^{\mathrm{p}}(r,\phi) = \sum_{k=1}^{3}[D_{k0}G_{ijk} + \bar{D}_{k0}\bar{G}_{ijk}] + r\sum_{k=1}^{3}[D_{k1}G_{ijk}\zeta_k + \bar{D}_{k1}\bar{G}_{ijk}\bar{\zeta}_k]$$
$$+\left\{r\sum_{k=1}^{3}C_{mnk1}\lambda_k\cos\phi + C_{mn33}A_1 + r[C_{mn33}\cos\phi A_2 + C_{mn33}\sin\phi A_3 + (C_{mn23}\cos\phi - C_{mn13}\sin\phi)A_4]\right\} l_{mi}l_{nj},$$

where l_{ij} denote the coordinate transformation matrix between (x_1, x_2, x_3) and $(\mathring{x}_1, \mathring{x}_2, x_3)$ given as

$$[l_{ij}] = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

H_{ik} and G_{ijk} are nothing but the vector and tensor components transformed from v_{ik} and $\tau_{ijk} = (C_{ijm1} + \mu_k C_{ijm2})v_{mk}$ to the components in the $\mathring{x}_i$ coordinate system, i.e. $H_{ik} = v_{mk}l_{mi}$, $G_{ijk} = \tau_{mnk}l_{mi}l_{nj}$.

APPENDIX B

In Appendix B, the expressions for $\mathbf{K}_c(\delta_n)$ and $\mathbf{b}_i$ (i = 1–4) in eqn (13) are presented in terms of the wedge angles and the material properties. Substitution of eqns (12a,b) into the near field conditions with the aid of the expressions in Appendix A yields

$$\sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n}[C_{kn}G_{ijk}(\alpha)\zeta_k^{\delta_n}(\alpha)+C_{(k-3)n}\bar{G}_{ijk}(\alpha)\bar{\zeta}_k^{\delta_n}(\alpha)] = -C_{mn33}A_1 l_{mi}(\alpha)l_{nj}(\alpha)$$

$$-r\left[\sum_{k=1}^{3} C_{mnk1}\lambda_k\cos\alpha + C_{mn33}\cos\alpha A_2 + C_{mn33}\sin\alpha A_3 + (C_{mn23}\cos\alpha - C_{mn13}\sin\alpha)A_4\right] l_{mi}(\alpha)l_{nj}(\alpha),$$

$$(ij = 12, 22, 23) \quad \text{(B1a)}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n}[C'_{kn}G'_{ijk}(\alpha)\zeta_k'^{\delta_n}(\alpha')+C'_{(k+3)n}\bar{G}'_{ijk}(\alpha')\bar{\zeta}_k'^{\delta_n}(\alpha')] = -C'_{mn33}A_1 l_{mi}(\alpha)l_{nj}(\alpha)$$

$$-r\left[\sum_{k=1}^{3} C'_{mnk1}\lambda'_k\cos\alpha' + C'_{mn33}\cos\alpha' A_2 + C'_{mn33}\sin\alpha' A_3 + (C'_{mn23}\cos\alpha' - C'_{mn13}\sin\alpha')A_4\right] l_{mi}(\alpha)l_{nj}(\alpha),$$

$$(ij = 12, 22, 23) \quad \text{(B1b)}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n}[(C_{kn}\tau_{ijk}+C_{(k+3)n}\bar{\tau}_{ijk})-(C'_{kn}\tau'_{ijk}+C'_{(k+3)n}\bar{\tau}'_{ijk})] = -(C_{ij33}-C'_{ij33})A_1$$

$$-r\left[\sum_{k=1}^{3}(C_{ijk1}\lambda_k - C'_{ijk}\lambda'_k) + (C_{ij33}-C'_{ij33})A_2 + (C_{ij23}-C'_{ij23})A_4\right], \quad (ij = 12, 22, 23) \quad \text{(B1c)}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{3} r^{\delta_n+1}[(C_{kn}v_{ik}+C_{(k+3)n}\bar{v}_{ik})-(C'_{kn}v'_{ik}+C'_{(k+3)n}\bar{v}'_{ik})]/(1+\delta_n) = -\tfrac{1}{2}(\lambda_i-\lambda'_i)r^2, \quad (i = 1, 2, 3) \quad \text{(B1d)}$$

where $G_{ijk}(\alpha)$, $l_{ij}(\alpha)$, $G'_{ijk}(\alpha')$ and $l_{ij}(\alpha')$ are values evaluated at $\phi = \alpha$ and $\phi = \alpha'$, respectively. To represent the right hand side of eqns (B1) in terms of the loading parameters A_1, A_2, A_3 and A_4 only, we write the solution λ_k in eqn (10) as

$$\lambda_k = \lambda_k^{(1)}A_2 + \lambda_k^{(2)}A_3 + \lambda_k^{(3)}A_4, \quad \text{(B2)}$$

where $\lambda_k^{(1)}$ (i = 1–3) are functions of material constants. Plugging eqn (B2) into eqns (B1), we can obtain the expressions for $\mathbf{K}_c(\delta_n)$ and $\mathbf{b}_i$ (i = 1–4) in eqn (13) in terms of the wedge angles and the material properties. Let $ij = 12$ for $p = 1$; $ij = 22$ for $p = 2$; $ij = 23$ for $p = 3$, then

$$[K_c(\delta_n)]_{pq} = G_{ijq}(\alpha)\zeta_q^{\delta_n}(\alpha), \quad [K_c(\delta_n)]_{p(q+3)} = \bar{G}_{ijq}(\alpha)\bar{\zeta}_q^{\delta_n}(\alpha), \quad [K_c(\delta_n)]_{p(q+6)} = [K_c(\delta_n)]_{p(q+9)} = 0,$$

$$[K_c(\delta_n)]_{(p+3)q} = [K_c(\delta_n)]_{(p+3)(q+3)} = 0, \quad [K_c(\delta_n)]_{(p+3)(q+6)} = G'_{ijq}(\alpha')\zeta_q'^{\delta_n}(\alpha'),$$

$$[K_c(\delta_n)]_{(p+3)(q+9)} = \bar{G}'_{ijq}(\alpha')\bar{\zeta}_q'^{\delta_n}(\alpha'), \quad [K_c(\delta_n)]_{(p+6)q} = \tau_{ijq},$$

$$[K_c(\delta_n)]_{(p+6)(q+3)} = \bar{\tau}_{ijq}, \quad [K_c(\delta_n)]_{(p-6)(q+6)} = -\tau'_{ijq}, \quad [K_c(\delta_n)]_{(p+6)(q+9)} = -\bar{\tau}'_{ijq},$$

$$[K_c(\delta_n)]_{(p+9)q} = v_{pq}, \quad [K_c(\delta_n)]_{(p+9)(q+3)} = \bar{v}_{pq}, \quad [K_c(\delta_n)]_{(p+9)(q+6)} = -v'_{pq}, \quad [K_c(\delta_n)]_{(p-9)(q+9)} = -\bar{v}'_{pq},$$

$$[b_1]_p = -C_{mn33}l_{mi}(\alpha)l_{nj}(\alpha), \quad [b_1]_{(p+3)} = -C'_{mn33}l_{mi}(\alpha')l_{nj}(\alpha'), \quad [b_1]_{(p+6)} = C'_{ij33}-C_{ij33}, \quad [b_1]_{(p+9)} = 0,$$

$$[b_2]_p = -\cos\alpha(B_{mn}^{(1)}+C_{mn33})l_{mi}(\alpha)l_{nj}(\alpha), \quad [b_2]_{(p-3)} = -\cos\alpha'(B_{mn}'^{(1)}+C'_{mn33})l_{mi}(\alpha')l_{nj}(\alpha'),$$

$$[b_2]_{(p+6)} = B_{ij}'^{(1)}+C'_{ij33}-B_{ij}^{(1)}-C_{ij33}, \quad [b_2]_{(p+9)} = 1/2(\lambda_p'^{(1)}-\lambda_p^{(1)}),$$

$$[b_3]_p = -(\cos\alpha B_{mn}^{(2)}+\sin\alpha C_{mn33})l_{mi}(\alpha)l_{nj}(\alpha),$$

$$[b_3]_{(p+3)} = -(\cos\alpha' B_{mn}'^{(2)}+\sin\alpha' C'_{mn33})l_{mi}(\alpha')l_{nj}(\alpha'),$$

$$[b_3]_{(p+6)} = B_{ij}'^{(2)}-B_{ij}^{(2)}, \quad [b_3]_{(p+9)} = 1/2(\lambda_p'^{(2)}-\lambda_p^{(2)}),$$

$$[b_4]_p = [\sin\alpha C_{mn13}-\cos\alpha(B_{mn}^{(3)}+C_{mn23})]l_{mi}(\alpha)l_{nj}(\alpha),$$

$$[b_4]_{(p+3)} = [\sin\alpha' C'_{mn13}-\cos\alpha'(B_{mn}'^{(3)}+C'_{mn23})]l_{mi}(\alpha')l_{nj}(\alpha'),$$

$$[b_4]_{(p+6)} = B_{ij}'^{(3)}+C'_{ij23}-B_{ij}^{(3)}-C_{ij23}, \quad [b_4]_{(p+9)} = 1/2(\lambda_p'^{(3)}-\lambda_p^{(3)}), \quad (p, q = 1, 2, 3)$$

where

$$B_{ij}^{(n)} = \sum_{k=1}^{3} C_{ijk1}\lambda_k^{(n)}, \quad B_{ij}'^{(n)} = \sum_{k=1}^{3} C'_{ijk1}\lambda_k'^{(n)} \quad (ij = 12, 22, 23,\ n = 1\text{–}3).$$